Back paper examination 2014
M.Math. II — Commutative algebra
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Throughout R stands for a commutative ring with $1 \neq 0$.

Group A

Answer any two questions from group A

Q 1.
Let x be a nilpotent element of the ring R. Show that $1 + x$ is a unit of R. Deduce from this that the sum of a nilpotent element and a unit is a unit in the ring.

Q 2.
Prove that for ideals I, J in a ring R, one has an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$.

Q 3.
Let I be an ideal of R and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

Group B

Answer any three questions from group B

Q 4.
Let R be a subring of S and S be integral over R.
(a) Show that if $x \in R$ is a unit in S then it is a unit in R.
(b) Show that the Jacobson radical of R is the contraction of the Jacobson radical of S.

Q 5.
Let $R \subset S$ be rings, and S is integral over R, and let p be a prime ideal of R. Then show that there exists a prime ideal q of S such that $q \cap R = p$.

Q 6.
Let R be an integral domain and K be its field of fractions. Show that the following are equivalent:
(a) R is a valuation ring of K.
(b) If I, J are any two ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

Q 7.
If R is a valuation ring of K and p is a prime ideal of R, then R_p and R/p are valuation rings of their field of fractions.

Group C

Answer any three questions from group C

Q 8.
Let R be a ring such that for each maximal ideal m of R, the local ring R_m is Noetherian, and for each $x \neq 0$ in R, the set of maximal ideals of R which contain x is finite. Show that R is Noetherian.

Q 9.
Let M be a Noetherian R module. Let $M[X]$ denote the set of all polynomials in X whose coefficients are in M. Show that $M[X]$ is a Noetherian $R[X]$ module.

Q 10.
Show that in an Artin ring every prime ideal is maximal.

Q 11.
Show that a ring R is Artin if and only if R is Noetherian with $\dim(R) = 0$.

Group D

Answer all questions from group D

Q 12.
Prove that the dimension of a Noetherian local ring is finite.

Q 13.
Let $(R, \mathfrak{m})$ be Noetherian local ring. Show that $\dim(R) \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. Dimension on the right hand side is the dimension of $(\mathfrak{m}/\mathfrak{m}^2)$ as a vector space over the residue field $R/\mathfrak{m}$.

Q 14.
Let (R, m) be a Noetherian local ring. Show that either R is a domain, or every principal prime ideal of R has ht 0.